

# Existence of sign-changing solutions for the nonlinear $p$ -Laplacian boundary value problem

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January 20, 2013

## Abstract

We study the nonlinear one-dimensional  $p$ -Laplacian equation

$$-(y'^{(p-1)})' + (p-1)q(x)y^{(p-1)} = (p-1)w(x)f(y) \text{ on } (0, 1),$$

with linear separated boundary conditions. We give sufficient conditions for the existence of solutions with prescribed nodal properties concerning the behavior of  $f(s)/s^{(p-1)}$  when  $s$  are at infinity and zero. These results are more general and complementary for previous known ones for the case  $p = 2$  and  $q$  is nonnegative.

AMS Subject Classification (2000) : 34A55, 34B24.

Keywords: Existence, zero, solutions, nonlinear,  $p$ -Laplacian, boundary value problem.

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# 1 Introduction

Consider the following nonlinear one-dimensional  $p$ -Laplacian equation,

$$-(y^{(p-1)})' + (p-1)q(x)y^{(p-1)} = (p-1)w(x)f(y) \text{ on } (0, 1), \quad (1.1)$$

with the separated boundary conditions

$$\begin{cases} S'_p(\alpha)y(0) - S_p(\alpha)y'(0) = 0, & \alpha \in [0, \pi_p), \\ S'_p(\beta)y(1) - S_p(\beta)y'(1) = 0, & \beta \in (0, \pi_p], \end{cases} \quad (1.2)$$

where  $p > 1$ ,  $y^{(p-1)} = |y|^{p-1} \operatorname{sgn} y = |y|^{p-2} y$ .

Denote by  $w = w(x) = S_p(x)$  the inverse function of the integral

$$x = \int_0^w \frac{dt}{(1-t^p)^{\frac{1}{p}}}, \text{ for } 0 \leq w \leq 1.$$

In 1979, Elbert [1] discussed the analogies between  $S_p(x)$  and  $\sin x$ . He showed that  $S_p(x)$  is the solution of

$$\begin{cases} (y^{(p-1)})' = -(p-1)y^{(p-1)}, \\ y(0) = 0, \quad y'(0) = 1, \end{cases} \quad (1.3)$$

and  $\pi_p \equiv \int_0^1 \frac{2dt}{(1-t^p)^{\frac{1}{p}}} = \frac{2\pi/p}{\sin(\pi/p)}$  is the first zero of  $S_p(x)$ . Furthermore, defining

$$S_p(x) = \begin{cases} S_p(\pi_p - x), & \text{if } \frac{\pi_p}{2} \leq x \leq \pi_p, \\ -S_p(x - \pi_p), & \text{if } \pi_p \leq x \leq 2\pi_p, \\ S_p(x - 2n\pi_p), & \text{for } n = \pm 1, \pm 2, \dots, \end{cases}$$

he obtained a sine-like function and the function  $S_p$  is so called the generalized sine function.

The application of the most original authors cited nowadays is the highly viscid fluid flow (cf. Ladyzhenskaya [2], Lions [3])

$$(BVP_1) \begin{cases} -\operatorname{div}((\nabla u)^{(p-1)}) + q(t)u^{(p-1)} = w(t)f(u), \\ u|_{\partial\Omega} = 0. \end{cases}$$

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This involves partial differential equations, but for symmetric flows, the ordinary differential operator (perhaps in radial form) is involved ( see, e.g., Binding & Drabek [4], del Pino, Elgueta & Manasevich [5], del Pino & Manasevich [6], Rabinowitz [7] and Walter [8]). It is well-known that the problem  $(BVP_1)$  has very similar properties as the classical case when  $p = 2$ , especially in the one-dimensional case (cf. Erbe [9], Kong [10], Lian, Wong & Yeh [11], Naito & Tanaka [12] and the reference therein). It has been investigated a good deal in the last twenty years or so under the general heading of  $p$ -Laplacian.

In 2000, Erbe [9] initiated the idea of connecting the equation

$$-y'' + q(t)y = w(t)f(y) \tag{1.4}$$

under the separated boundary value conditions with the Sturm-Liouville eigenvalue problem (SLEP). Using the fixed point index method and comparing the values of  $\frac{f(s)}{s}$ ,  $s \in (0, \infty)$ , with the smallest eigenvalue of the corresponding (SLEP), the existence of positive solutions of (1.4) was established. But, due to the limitation of the approach, he only discussed the case  $q > 0$  and certain boundary conditions and nothing was found for the existence of solutions with zeros in  $(0, 1)$ .

Next, Naito & Tanaka [12] compared the equation with the  $k$ -th eigenvalue of the linear equation, and applied the method of energy function and the Sturm-Picone comparison theorem to establish the sufficient conditions for the existence of multiple solutions with prescribed numbers of zeros in  $(0, 1)$  for the case  $q \equiv 0$ , i.e.,

$$\begin{cases} u'' + w(t)f(u) = 0 & \text{on } (0, 1), \\ u(0) = u(1) = 0. \end{cases}$$

Recently, Kong [10] generalized the results in [12] to the case with nonzero  $q$ . This extension is not trivial due to the fact that the energy function with a general  $q$  may not be nonnegative. He also obtained results on the nonexistence of certain types of solutions.

When  $\frac{f(s)}{s}$ ,  $s \in (0, \infty)$ , is between  $f_0$  and  $f_\infty$  (see (C3) below), the conditions for the existence and nonexistence become necessary and sufficient.

In 2008, Naito & Tanaka [13] studied the Dirichlet quasi-linear differential equation with  $q \equiv 0$  by the shooting method together with the qualitative theory. In the same year, Lee & Sim [14] considered the same case,  $q \equiv 0$ , with Dirichlet boundary conditions with a nonnegative measurable function  $w(x)$  on  $(0, 1)$  which may be singular at  $x = 0$  and/or  $x = 1$ . They gave the global analysis for sign-changing solutions employing a bifurcation argument.

The aim of this paper is to establish the sufficient condition for the existence of solutions of the BVP (1.1)-(1.2) with prescribed numbers of zeros in terms of the ratio  $f(s)/s^{(p-1)}$  at infinity and zero. Inspired by the ideas of [10, 13, 14], we study the BVP (1.1)-(1.2) with a nonzero  $q$  and general separated boundary conditions. We employ the generalized Prüfer substitution and comparison theorem in our arguments. Our results generalize partly those ones in Kong [10] and Naito & Tanaka [13].

In this paper, we assume the following conditions hold:

(C1)  $q, w \in C^1[0, 1]$  and  $w > 0$  on  $[0, 1]$ ;

(C2)  $f \in C(\mathbb{R})$ ,  $f(s) > 0$  for  $s > 0$ ,  $f(-s) = -f(s)$  for  $s > 0$ , and  $f$  is locally Lipschitz continuous on  $(0, \infty)$ ;

(C3) there exist  $0 \leq f_0, f_\infty \leq \infty$  such that  $\lim_{s \rightarrow 0^+} \frac{f(s)}{s^{p-1}} = f_0$  and  $\lim_{s \rightarrow \infty} \frac{f(s)}{s^{p-1}} = f_\infty$ .

The typical examples satisfying (C2)-(C3) are

- (i)  $f$  is a finite sum of sign power functions,  $f(s) = \sum_{i=1}^n A_i s^{(\ell_i)}$ , with  $A_i, \ell_i > 0$ ;
- (ii)  $f$  is an odd exponential function such as  $f(s) = A \sinh Bs$ , where  $A, B > 0$ .

In order to discuss our results, we compare the BVP (1.1) - (1.2) with the  $p$ -Laplacian eigenvalue problem

$$-(y'^{(p-1)})' = (p-1)(\lambda w(x) - q(x))y^{(p-1)} \text{ on } (0, 1), \quad (1.5)$$

coupled with the boundary condition (1.2). It is known that BVP (1.5) and (1.2) has a countable number of eigenvalues  $\lambda_i$  satisfying

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_n < \dots \rightarrow \infty ,$$

and the eigenfunction corresponding to  $\lambda_n$  has  $n$  zeros in  $(0, 1)$  (cf. Binding & Drabek [4], Reichel & Walter [15]). The followings are our main results.

**Theorem 1.1.** *(i) For all  $y \in (0, \infty)$ , if  $\frac{f(y)}{y^{p-1}} < \lambda_n$  for some  $n$ , then BVP (1.1) -(1.2) has no solution with exactly  $i$  zeros in  $(0, 1)$  for any  $i \geq n$ .*

*(ii) For all  $y \in (0, \infty)$ , if  $\lambda_n < \frac{f(y)}{y^{p-1}}$  for some  $n$ , then BVP (1.1) -(1.2) has no solution with exactly  $i$  zeros in  $(0, 1)$  for any  $i \leq n$ .*

*(iii) For all  $y \in (0, \infty)$ , if  $\frac{f(y)}{y^{p-1}} \neq \lambda_n$  for any  $n$ , then BVP (1.1) -(1.2) has no nontrivial solution.*

**Theorem 1.2.** *Assume there exists  $n$  such that either  $\lambda_n \in (f_0, f_\infty)$  or  $(f_\infty, f_0)$ . Then BVP (1.1) -(1.2) has a solution with exactly  $n$  zeros in  $(0, 1)$ .*

The combination of Theorems 1.1 and 1.2 leads to the following.

**Corollary 1.3.** *Assume  $\frac{f(y)}{y^{p-1}} \in (f_0, f_\infty)$  or  $(f_\infty, f_0)$  for all  $y \in (0, \infty)$ . Then for some  $n$  such that  $\lambda_n > 0$ , BVP (1.1) -(1.2) has a solution with exactly  $n$  zeros in  $(0, 1)$  if and only if  $\lambda_n \in (f_0, f_\infty)$  or  $(f_\infty, f_0)$ .*

This paper is organized as follows. After the introduction, we establish the global existence and uniqueness of the solutions of the initial value problems associated with (2.1) in Section 2. In Section 3, we give some technical lemmas for the proof of Theorem 1.2. We give the proofs of the main theorems in Section 4. In the appendix, we give the proof of Proposition 2.2 represented in Section 2.

## 2 Results on initial value problems

We first establish the basic properties of the system consisting of (1.1) with the initial condition

$$y(0) = \eta_1, \quad y'(0) = \eta_2, \quad (2.1)$$

where  $\eta_1, \eta_2 \in \mathbb{R}$  is a parameter. It is known the solution of this IVP, if it exists, may not be unique under a general condition (cf. Walter [8, p.181] and Reichel & Walter [16, Theorem 4] for the radial case). Now we show that the global existence and uniqueness are guaranteed under the conditions (C1)-(C3). First motivated by [10], we introduce a generalized energy function  $E[y](x)$  to derive the global existence.

**Proposition 2.1.** *For any  $\eta_1, \eta_2 \in \mathbb{R}$ , the IVP (1.1) and (2.1) has a solution  $y(x)$  which exists over the whole interval  $[0, 1]$ .*

*Proof.* Note that the IVP (1.1) and (2.1) can be written by

$$\begin{cases} y'(x) = z(x)^{(p^*-1)}, \\ z'(x) = (p-1)q(x)y(x)^{(p-1)} - (p-1)w(x)f(y(x)), \end{cases} \quad (2.2)$$

with  $y(0) = \eta_1$  and  $z(0) = \eta_2^{(p-1)}$  where  $p^* = \frac{p}{p-1}$  is the conjugate exponent of  $p$ . Then the local existence of a solution is guaranteed by Peano existence theorem. We will divide the proof into two steps to prove the global existence.

(i) Consider the case  $f_\infty = \infty$ . Suppose  $y(x)$  does not exist on the whole interval  $[0, 1]$ .

Without loss of generality, we may assume  $y(x)$  exists on a maximal right interval  $[0, c)$  for some  $c \in (0, 1)$ . Then  $y(x)$  is unbounded on  $[0, c)$ . For otherwise, if  $y(x)$  is bounded on  $[0, c)$ , then, from (1.1),  $\lim_{x \rightarrow c} y'(x)$  is bounded. This implies  $y(x)$  can be extended through  $c$ . This contradicts that  $[0, c)$  is the maximal right interval of existence for  $y(x)$ . Now define the generalized energy function for the solution  $y$  as

$$E[y](x) = \frac{|y'(x)|^p}{p} - \frac{1}{p}q(x)|y(x)|^p + w(x)F(y(x)), \quad (2.3)$$

where  $F(y) = \int_0^y f(s)ds$ . Then (1.1) implies that

$$E[y]'(x) = -\frac{1}{p}q'(x)|y(x)|^p + \frac{w'(x)}{w(x)}w(x)F(y(x)).$$

Let  $k = \max\{\frac{|w'(x)|}{w(x)} : x \in [0, 1]\}$ . Then

$$E[y]'(x) \leq -\frac{k+1}{p}q(x)|y(x)|^p + \frac{1}{p}[(k+1)q(x) - q'(x)]|y(x)|^p + kw(x)F(y(x)). \quad (2.4)$$

Because  $w > 0$  is continuous and  $q, q'$  are bounded on  $[0, 1]$ , we can find a constant  $h > 0$  such that

$$\frac{h}{p}[(k+1)q(x) - q'(x)] \leq w(x) \quad \text{and} \quad \frac{h}{p}|q(x)| \leq w(x). \quad (2.5)$$

Since  $f_\infty = \infty$ , there exists  $M > 0$  such that  $|y|^p \leq hF(y)$  for  $|y| \geq M$ . Define  $I_1 = \{x \in [0, c) : |y(x)| \leq M\}$  and  $I_2 = \{x \in [0, c) : |y(x)| > M\}$ . Then from (2.4), there exists  $N > 0$  such that  $E'[y](x) \leq N$  for  $x \in I_1$ , and for  $x \in I_2$

$$E[y]'(x) \leq (k+1)\left[-\frac{1}{p}q(x)|y(x)|^p + w(x)F(y(x))\right] \leq (k+1)E[y](x).$$

Hence, from the second inequality in (2.5) that  $E[y](x) \geq 0$  on  $I_2$ ; we have

$$E[y]'(x) \leq N + (k+1)E[y](x), \quad x \in [0, c).$$

Integrating both sides of the above inequality, we get that for  $x \in [0, c)$ ,

$$E[y](x) \leq E[y](0) + N + \int_0^x (k+1)E[y](t)dt.$$

By the Gronwall inequality,

$$E[y](x) \leq (E[y](0) + N) \exp[(k+1)x], \quad \text{for } x \in [0, c).$$

Therefore,

$$\limsup_{x \rightarrow c-} E[y](x) < \infty. \quad (2.6)$$

On the other hand, since  $y(x)$  is unbounded on  $[0, c)$ , there exists a sequence  $t_n \rightarrow c-$  such that  $|y(t_n)| \rightarrow \infty$ . Hence  $\lim_{n \rightarrow \infty} \frac{F(y(t_n))}{|y(t_n)|^p} = \frac{f_\infty}{p} = \infty$ . By (2.3),

$$E[y](t_n) \geq \left(-\frac{1}{p}q(t_n) + w(t_n)\frac{F(y(t_n))}{|y(t_n)|^p}\right)|y(t_n)|^p \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

This contradicts with (2.6).

(ii) Let  $f_\infty < \infty$ . Integrating (2.2) over any compact interval  $[0, k] \subset [0, \infty)$ , we obtain

$$y(x) = \eta_1 + \int_0^x z(t)^{(p^*-1)}dt, \quad (2.7)$$

$$z(x) = \eta_2^{(p-1)} + (p-1) \int_0^x q(t)y(t)^{(p-1)}dt - (p-1) \int_0^x w(t)f(y(t))dt, \quad (2.8)$$

for any  $x \in [0, k]$ . Note that  $p^* - 1 = \frac{1}{p-1}$ . Since  $f_\infty < \infty$ , we have

$$|f(y)| \leq c_1|y|^{p-1} \quad \text{for } |y| \geq M',$$

where  $c_1$  and  $M'$  are some positive numbers. For  $|y| < M'$ , it is easy to obtain the boundedness of  $z(x)$  by (2.8). So, for  $|y| \geq M'$  and  $x \in [0, k]$ , it follows from Hölder inequality that

$$|z(x)| \leq c_2 + c_3 \left( \int_0^x |y(t)|^p dt \right)^{\frac{p-1}{p}}.$$



i.e.,

$$|z(x)|^{\frac{p}{p-1}} \leq c_2^{\frac{p}{p-1}} + \left( \int_0^x |y(t)|^p dt \right) (c_3^{\frac{p}{p-1}} + O(1)).$$

Thus

$$|z(x)|^{\frac{p}{p-1}} \leq c_4 + c_5 \int_0^x |y(t)|^p dt, \quad \text{for } x \in [0, k]. \quad (2.9)$$

Similar arguments, we can obtain

$$|y(x)|^p \leq c_6 + c_7 \int_0^x |z(t)|^{p^*} dt, \quad \text{for } x \in [0, k]. \quad (2.10)$$

From (2.9) and (2.10), we have

$$|y(x)|^p + |z(x)|^{p^*} \leq c(k; p) + d(k; p) \int_0^x (|y(t)|^p + |z(t)|^{p^*}) dt,$$

where  $c(k; p)$  and  $d(k; p)$  are some positive constants depending on  $p$  and  $k$ . By Gronwall inequality,

$$|y(x)|^p + |z(x)|^{p^*} \leq c(k; p) \exp[d(k; p)x] < \infty.$$

Therefore, the solution exists over the whole interval  $[0, 1]$ . □

Following the ideas of [8], [10] and [13], we can prove the following uniqueness of the solution of the IVP, which will be proven in the appendix.

**Proposition 2.2.** *For any  $\eta_1, \eta_2 \in \mathbb{R}$ , the solution  $y(x; \eta_1, \eta_2)$  of the IVP (1.1) and (2.1) is unique on  $[0, 1]$ . Furthermore,  $y(x; \eta_1, \eta_2)$  and  $y'(x; \eta_1, \eta_2)$  are continuous in  $(x; \eta_1, \eta_2) \in [0, 1] \times \mathbb{R}^2$ .*

### 3 Some technical lemmas

In this section, we will derive three lemmas for the proof of main theorems. First we consider the IVP consisting (1.1) with the initial condition

$$y(0) = \rho S_p(\alpha), \quad y'(0) = \rho S'_p(\alpha), \quad (3.1)$$

where  $\rho > 0$  is a parameter. Denote by  $y(x; \rho)$  the solution of (1.1) and (3.1). Consider the modifier Prüfer substitution

$$y(x; \rho) = r(x; \rho)S_p(\theta(x; \rho)) , \quad y'(x; \rho) = r(x; \rho)S'_p(\theta(x; \rho)) .$$

Then we have  $\theta(0; \rho) = \alpha, r(x; \rho) = (|y(x; \rho)|^p + |y'(x; \rho)|^p)^{1/p} > 0$  and

$$\theta'(x; \rho) = |S'_p(\theta(x; \rho))|^p + \frac{w(x)f(y(x; \rho))S_p(\theta(x; \rho))}{r(x; \rho)^{p-1}} - q(x)|S_p(\theta(x; \rho))|^p , \quad (3.2)$$

$$r'(x; \rho) = S'_p(\theta(x; \rho)) \left[ (1 + q(x))r(x; \rho)S_p^{(p-1)}(\theta(x; \rho)) - \frac{w(x)f(y(x; \rho))}{r(x; \rho)^{p-2}} \right] . \quad (3.3)$$

Similarly, the Prüfer angle  $\phi_n$  for (1.5) and (3.1) with  $\lambda = \lambda_n$  satisfies

$$\phi'_n(x; \rho) = |S'_p(\phi_n(x; \rho))|^p + [\lambda_n w(x) - q(x)]|S_p(\phi_n(x; \rho))|^p . \quad (3.4)$$

**Lemma 3.1.** (a) Assume  $f_0 < \lambda_n$  for some  $n$ . Then there exists a sufficiently small  $\rho_*$  such that  $\theta(1; \rho) < n\pi_p + \beta$  for all  $\rho \in (0, \rho_*)$ .

(b) Assume  $f_0 > \lambda_n$  for some  $n$ . Then there exists a sufficiently small  $\rho_*$  such that  $\theta(1; \rho) > n\pi_p + \beta$  for all  $\rho \in (0, \rho_*)$ .

*Proof.* We give the proof of (a) here. The proof of part (b) is similar and will be omitted.

Since  $f_0 < \lambda_n < \infty$ ,  $\frac{f(y)}{y^{(p-1)}}$  can be continuously extended to  $y = 0$  and there exists  $\delta > 0$  such that

$$\frac{f(y)}{y^{(p-1)}} < \lambda_n \quad \text{for} \quad |y| < \delta .$$

Since  $y \equiv 0$  is a solution of (1.1), by the continuous dependence of solutions on the initial conditions, there exists  $\rho_* > 0$  such that  $|y(x; \rho)| < \delta$  for  $\rho < \rho_*$  and  $x \in [0, 1]$ . From (3.2), for  $\rho < \rho_*$  and  $x \in [0, 1]$ , we have

$$\theta'(x; \rho) < |S'_p(\theta(x; \rho))|^p + [\lambda_n w(x) - q(x)]|S_p(\theta(x; \rho))|^p .$$

Let  $u_n(x; \rho)$  be the solution of the IVP (1.5) and (3.1) with  $\lambda = \lambda_n$  and let  $\phi_n$  be its Prüfer angle. Then  $u_n(x; \rho)$  is an eigenfunction of the BVP (1.5) and (1.2) corresponding to the

eigenvalue  $\lambda = \lambda_n$ ; thus  $\phi_n(1; \rho) = n\pi_p + \beta$ . By the comparison theorem, we obtain that  $\theta(1; \rho) < \phi_n(1; \rho)$ . This completes the proof.  $\square$

**Lemma 3.2.** *For  $M > 0$  and  $\rho > 0$  define  $I_{M,\rho} = \{x \in [0, 1] : |y(x; \rho)| < M\}$ . Then for any  $M, L > 0$ , there exists a sufficiently large  $\rho^* > 0$  such that  $|y'(x; \rho)| > L$  for  $\rho > \rho^*$  and  $x \in I_{M,\rho}$ .*

*Proof.* (i) Let  $f_\infty < \infty$ . For  $\rho > 0$  and from (3.3), it is easy to find  $K_1 > 0$  such that

$$r'(x; \rho) \geq -K_1 \quad \text{for } x \in I_{M,\rho}. \quad (3.5)$$

Since  $f_\infty < \infty$ , there exists  $K_2 > 0$  such that  $|\frac{f(y(x; \rho))}{y(x; \rho)^{(p-1)}}| \leq K_2$  for  $x \in [0, 1] \setminus I_{M,\rho}$ . By (3.3), we have, for  $x \in [0, 1] \setminus I_{M,\rho}$ ,

$$\begin{aligned} r'(x; \rho) &= r(x; \rho) S'_p(\theta(x; \rho)) S_p^{(p-1)}(\theta(x; \rho)) \left[ 1 + q(x) - \frac{w(x)f(y(x; \rho))}{y^{(p-1)}(x; \rho)} \right] \\ &\geq -r(x; \rho) [|1 + q(x)| + K_2 w(x)] \\ &\geq -K_3 r(x; \rho), \end{aligned} \quad (3.6)$$

where  $K_3 = \max\{|1 + q(x)| + K_2 w(x) : x \in [0, 1]\}$ . Combining (3.5) and (3.6),

$$r'(x; \rho) \geq -K_1 - K_3 r(x; \rho) \quad \text{for } \rho > 0 \text{ and } x \in [0, 1].$$

Solving the above linear differential inequality for  $x \in [0, 1]$ , we have

$$r(x; \rho) \geq -\frac{K_1}{K_3} + \left(\rho + \frac{K_1}{K_3}\right) e^{-K_3 x} \rightarrow \infty \quad \text{as } \rho \rightarrow \infty$$

uniformly in  $[0, 1]$ . Therefore, for any  $L > 0$ , there exists  $\rho^* > 0$  such that  $\rho > \rho^*$  and  $x \in I_{M,\rho}$

$$M^p + |y'(x; \rho)|^p \geq r(x; \rho) > M^p + L^p.$$

This leads to have that  $|y'(x; \rho)| > L$ .

(ii) Let  $f_\infty = \infty$ . Recall the generalized energy function  $E[y](x; \rho)$  defined as (2.3),

$$E[y](x; \rho) = \frac{|y'(x; \rho)|^p}{p} - \frac{1}{p}q(x)|y(x; \rho)|^p + w(x)F(y(x; \rho)) \quad (3.7)$$

where  $F(y) = \int_0^y f(s)ds$ . Then, letting  $k = \max\{\frac{|w'(x)|}{w(x)} : x \in [0, 1]\}$ ,

$$\begin{aligned} E[y]'(x; \rho) &= -\frac{1}{p}q'(x)|y(x; \rho)|^p + w'(x)F(y(x; \rho)) \\ &\geq \frac{k+1}{p}q(x)|y(x; \rho)|^p - \frac{1}{p}[(k+1)q(x) + q'(x)]|y(x; \rho)|^p \\ &\quad - kw(x)F(y(x; \rho)). \end{aligned} \quad (3.8)$$

Because  $w > 0$  is continuous and  $q, q'$  are bounded on  $[0, 1]$ , we can find a constant  $h > 0$  such that

$$\frac{h}{p}|(k+1)q(x) + q'(x)| \leq w(x) \quad \text{and} \quad \frac{h}{p}|q(x)| \leq w(x). \quad (3.9)$$

Since  $f_\infty = \infty$ , when  $M$  is sufficiently large, we have  $|y|^p \leq hF(y)$  for all  $|y| \geq M$ . Then from (3.8), when  $x \in I_{M, \rho}$ ,  $E[y]'(x; \rho)$  is bounded below. But for  $x \in [0, 1] \setminus I_{M, \rho}$ , by (3.9),  $E[y]'(x; \rho) \geq -(k+1)E[y](x; \rho)$ . Hence, for all  $x \in [0, 1]$ ,

$$E[y]'(x; \rho) \geq -N - (k+1)E[y](x; \rho).$$

Solving the above linear differential inequality for  $x \in [0, 1]$ , we obtain

$$E[y](x; \rho) \geq -\frac{N}{k+1} + (E[y](0; \rho) + \frac{N}{k+1}) \exp[-(k+1)x]. \quad (3.10)$$

Note that, from the initial condition (3.1),

$$E[y](0; \rho) = \rho^p \left[ \frac{|S_p'(\alpha)|^p}{p} - \frac{q(x)|S_p(\alpha)|^p}{p} + w(x) \frac{F(\rho S_p(\alpha))}{\rho^p} \right].$$

When  $\alpha = 0$ ,

$$\lim_{\rho \rightarrow \infty} E[y](0; \rho) = \lim_{\rho \rightarrow \infty} \frac{\rho^p}{p} = \infty,$$

and when  $\alpha \in (0, \pi_p)$ , it follows from  $f_\infty = \infty$  that

$$\lim_{\rho \rightarrow \infty} \frac{F(\rho S_p(\alpha))}{\rho^p} = \lim_{y \rightarrow \infty} \frac{F(y)}{y^p} |S_p(\alpha)|^p = \infty.$$

So we have  $\lim_{\rho \rightarrow \infty} E[y](0; \rho) = \infty$ . Therefore, from (3.10) we get that

$$\lim_{\rho \rightarrow \infty} E[y](x; \rho) = \infty \quad \text{uniformly for } x \in [0, 1]. \quad (3.11)$$

Note that, in (3.7), the term  $|\frac{1}{p}q(x)|y(x; \rho)|^p + w(x)F(y(x; \rho))|$  is uniformly bounded for all  $\rho > 0$  and  $x \in I_{M, \rho}$ . So, by (3.7) and (3.11), we may choose  $\rho^*$  such that  $\rho > \rho^*$  and  $x \in I_{M, \rho}$ ,  $|y'(x; \rho)| > L$ .

□

**Lemma 3.3.** (a) Assume  $f_\infty > \lambda_n$  for some  $n$ . Then there exists a sufficiently large  $\rho^*$  such that  $\theta(1; \rho) > n\pi_p + \beta$  for all  $\rho \in (\rho^*, \infty)$ .

(b) Assume  $f_\infty < \lambda_n$  for some  $n \in \mathbb{N}_k$ . Then there exists a sufficiently large  $\rho^*$  such that  $\theta(1; \rho) < n\pi_p + \beta$  for all  $\rho \in (\rho^*, \infty)$ .

*Proof.* We give the proof of (a) here. The proof of part (b) is similar.

Assume the contrary. Then there exists  $\rho_l$  with  $\rho_l \rightarrow \infty$  such that  $\theta(1; \rho_l) \leq n\pi_p + \beta$ . This implies that  $y(x; \rho_l)$  has at most  $n$  zeros in  $(0, 1)$ . Since  $f_\infty > \lambda_n$ , we can choose  $\lambda > 0$  such that  $\lambda_n < \lambda < f_\infty$  and take  $M > 0$  so that

$$\frac{f(y(x; \rho))}{y(x; \rho)^{(p-1)}} \geq \lambda \quad \text{for } |y(x; \rho)| \geq M.$$

We divide the proof into several steps:

- (i) We claim that the measure of  $I_{M, \rho}$  tends to zero as  $\rho = \rho_l \rightarrow \infty$ . It is easy to see that for each  $\rho = \rho_l$ ,  $I_{M, \rho} \cap (0, 1)$  is an open set and hence is a union of disjoint intervals in  $(0, 1)$ , i.e.,

$$I_{M, \rho} \cap (0, 1) = \cup_{i=1}^j (a_i, b_i),$$

where  $0 \leq a_i < b_i \leq 1$ . If  $0 < a_i$  and  $b_i < 1$ , by Lemma 3.2, for  $\rho = \rho_l$  sufficiently large,  $y(x; \rho)$  is monotone on  $[a_i, b_i]$ , and hence  $|y(a_i; \rho)| = |y(b_i; \rho)| = M$  and  $y(a_i; \rho)y(b_i; \rho) < 0$ . This implies that  $(a_i, b_i)$  contains exactly one zero of  $y(x; \rho)$ , so  $j \leq n+2$ . Applying Lemma 3.2 again, for any  $L > 0$  there exists  $\rho(L) > 0$  such that if  $\rho = \rho_l > \rho(L)$ , then  $y'(x; \rho)$  has the same sign and  $|y'(x; \rho)| > L$  in  $(a_i, b_i)$  for each  $1 \leq i \leq j$ . Thus,

$$2M = |y(b_i; \rho) - y(a_i; \rho)| = \left| \int_{a_i}^{b_i} y'(t; \rho) dt \right| > L(b_i - a_i).$$

i.e.,  $b_i - a_i < \frac{2M}{L}$ . So,  $\|I_{M, \rho}\| \leq \frac{2(n+2)M}{L}$ , where  $\|\cdot\|$  is the Lebesgue measure. Therefore,

$$\lim_{\rho_l \rightarrow \infty} \|I_{M, \rho_l}\| = 0. \quad (3.12)$$

(ii) Next, we try to reach a contradiction to our assumption. For each  $\rho = \rho_l$ , let  $\phi(x; \rho)$  and  $\phi_n(x; \rho)$  be the Prüfer angles of the solution of (1.5) and (3.1) with  $\lambda$  and  $\lambda_n$ , respectively. Then  $\phi_n(1; \rho) = n\pi_p + \beta$  and hence, by the comparison theorem,  $\phi(1; \rho) = n\pi_p + \beta + \epsilon$  for some  $\epsilon > 0$ . Recall that  $\phi(x; \rho)$  satisfies

$$\phi'(x; \rho) = |S'_p(\phi(x; \rho))|^p + [\lambda w(x) - q(x)]|S_p(\phi(x; \rho))|^p \equiv G(x, \rho, \phi), \quad (3.13)$$

and  $\phi(0; \rho) = \alpha$ . On the other hand, define

$$g(x; \rho) = \begin{cases} \frac{f(y(x; \rho))}{y(x; \rho)^{(p-1)}}, & |y(x; \rho)| < M \\ \lambda, & |y(x; \rho)| \geq M. \end{cases}$$

and let  $\theta(x; \rho)$  be the Prüfer angle of (1.1) and (3.1). Then, by (3.2),

$$\theta'(x; \rho) \geq |S'_p(\theta(x; \rho))|^p + w(x)g(x; \rho)|S_p(\theta(x; \rho))|^p - q(x)|S_p(\theta(x; \rho))|^p \equiv F(x, \rho, \theta).$$

Let  $\bar{\theta}(x; \rho)$  be the solution of the equation,

$$\bar{\theta}'(x; \rho) = F(x, \rho, \bar{\theta}) \quad (3.14)$$

satisfying  $\bar{\theta}(0; \rho) = \alpha$ . From (3.13) and (3.14) we have that for  $\rho = \rho_l$  and  $x \in [0, 1]$ ,

$$\begin{aligned}
\bar{\theta}(x; \rho) - \phi(x; \rho) &= \int_0^x (F(s, \rho, \bar{\theta}) - G(s, \rho, \phi)) ds \\
&= \int_0^x [(F(s, \rho, \bar{\theta}) - G(s, \rho, \bar{\theta})) + (G(s, \rho, \bar{\theta}) - G(s, \rho, \phi))] ds \\
&= \int_0^x w(s) [g(s; \rho) - \lambda] |S_p(\bar{\theta}(s; \rho))|^p ds \\
&\quad + \int_0^x \frac{\partial}{\partial \phi} G(s, \rho, \zeta) [\bar{\theta}(s; \rho) - \phi(s; \rho)] ds
\end{aligned} \tag{3.15}$$

where  $\zeta(s; \rho)$  is between  $\bar{\theta}(s; \rho)$  and  $\phi(s; \rho)$ . Since  $g(x; \rho) = \lambda$  for  $x \in [0, 1] \setminus I_{M, \rho}$  and  $g(x; \rho)$  is continuous on  $I_{M, \rho}$ , we have, by (3.12),

$$\left| \int_0^x w(s) [g(s; \rho) - \lambda] |S_p(\bar{\theta}(s; \rho))|^p ds \right| \leq \int_{I_{M, \rho}} w(s) |g(s; \rho) - \lambda| ds \rightarrow 0 \tag{3.16}$$

as  $\rho = \rho_l \rightarrow \infty$ . Note that  $|\frac{\partial}{\partial \phi} G(x, \rho, \phi)|$  is uniformly bounded by some  $K > 0$  for all  $x \in [0, 1]$ . Thus, by (3.15) and (3.16), for any  $\delta > 0$  there exists a large  $\rho^*$  such that for  $\rho \in (\rho^*, \infty)$ ,

$$|\bar{\theta}(x; \rho) - \phi(x; \rho)| < \delta + \int_0^x K |\bar{\theta}(s; \rho) - \phi(s; \rho)| ds.$$

By Gronwall inequality, we have

$$|\bar{\theta}(x; \rho) - \phi(x; \rho)| < \delta e^{Kx} < \epsilon$$

if  $\delta < \epsilon e^{-K}$ . Hence,  $\bar{\theta}(x; \rho) > \phi(x; \rho) - \epsilon$  on  $[0, 1]$ . Furthermore,

$$\theta(1; \rho) \geq \bar{\theta}(1; \rho) > \phi(1; \rho) - \epsilon = n\pi_p + \beta.$$

This contradicts with our assumption. So the proof is completed.

□

## 4 Proofs of Theorem 1.1 & 1.2

*Proof of Theorem 1.1.* (i) Assume the contrary that there exists a solution  $y(x)$  of (1.1)-(1.2) with exactly  $i$  zeros in  $(0, 1)$  for some  $i \geq n$ . Let  $\bar{w}(x) = w(x) \frac{f(y(x))}{y(x)^{(p-1)}}$ . Then  $\bar{w}(x)$  is continuous on  $[0, 1]$  since  $f_0 < \infty$ . Denote that  $\theta(x)$  is the Prüfer angle of  $y(x)$ . Then  $\theta(x)$  satisfies (3.2) and  $\theta(1) = (i + 1)\pi_p$ ,  $i \geq n$ . Note that  $\frac{f(y(x))}{y(x)^{(p-1)}} \neq \infty$  in this case, and

$$w(x) \frac{f(y(x))S_p(\theta(x))}{r(x)^{(p-1)}} = \frac{w(x)f(y(x))}{y(x)^{(p-1)}} |S_p(\theta(x))|^p < \lambda_n w(x) |S_p(\theta(x))|^p.$$

By the comparison theorem, we find that  $\theta(1) < \phi_n(1) = (n + 1)\pi_p$ . This reaches a contradiction.

(ii) Assume the contrary that (1.1)-(1.2) has a solution  $y(x)$  with exactly  $i$  zeros in  $(0, 1)$  for some  $i \leq n$ . By similar argument as the above, we have that  $\theta'(x) > \phi'_n(x)$  a.e. on  $[0, 1]$ . By the comparison theory, we obtain that  $\theta(1) > \phi_n(1) = (n + 1)\pi_p$ .

(iii) The assumption implies that either

$$(a) \quad \lambda_n < \frac{f(y(x))}{y(x)^{p-1}} < \lambda_{n+1} \text{ for some } n \text{ for all } y \in (0, \infty)$$

or

$$(b) \quad 0 < \frac{f(y(x))}{y(x)^{p-1}} < \lambda_0 \text{ for all } y \in (0, \infty) \text{ if } k = 0.$$

By the uniqueness, we have that the number of zeros of  $y(x)$  is finite. Then the conclusion follows from (i) and (ii) immediately.

□

*Proof of Theorem 1.2.* Here we consider the case of  $f_0 < \lambda_n < f_\infty$ . By Lemma 3.1, there exists  $\rho_* > 0$  such that  $\theta(1; \rho) < n\pi_p + \beta$  for all  $\rho \in (0, \rho_*)$ . By Lemma 3.3, there exists  $\rho^* > \rho_*$  such that  $\theta(1; \rho) > n\pi_p + \beta$  for all  $\rho \in (\rho^*, \infty)$ . By continuous dependence on



parameters, there exists  $\rho_n \in [\rho_*, \rho^*]$  such that  $\theta(1; \rho_n) = n\pi_p + \beta$ . This implies that  $y(x; \rho_n)$  is a solution of the BVP (1.1)-(1.2) with exactly  $n$  zeros in  $(0, 1)$ .

The other case is similar by applying Lemma 3.1(b) and 3.3 (b).

□

## Appendix. The proof of Proposition 2.2

In the proof, we need the following lemma ([8], p180).

**Lemma A.** *Let  $W \in C^1(I)$ ,  $x_0 \in I$  and  $W(x_0) = 0$ , where  $I$  is a compact interval containing  $x_0$ . Denote  $\|W\|_x$  the maximum of  $W$  in the interval from  $x_0$  to  $x$ . Then*

$$|W'(x)| \leq K\|W\|_x \quad \text{in } I \text{ implies } W = 0 \quad \text{for } |x - x_0| \leq \frac{1}{K}, \quad x \in I. \quad (4.1)$$

Now, it suffices to show the uniqueness of a local solution of the IVP. We will divide the proof into the following cases.

(i) Let  $\eta_1 \cdot \eta_2 \neq 0$ . Since the right hand side of (2.2) is locally Lipschitz continuous in  $y$ ,  $z \in \mathbb{R} \setminus \{0\}$ , the local solution of (1.1) and (2.1) is unique for this case.

(ii) Let  $\eta_1 = \eta_2 = 0$ . In this case we apply two versions of energy functions to prove the solution  $y(x; \eta_1, \eta_2) \equiv 0$ .

(a) Let  $f_0 = \infty$ . Let  $E[y](x)$  be defined as (2.3). Similar arguments in the proof of Proposition 2.1, we have

$$E[y]'(x) \leq -\frac{(k+1)}{p}q(x)|y(x)|^p + \frac{1}{p}[(k+1)q(x) - q'(x)]|y(x)|^p + kw(x)F(y(x)), \quad (4.2)$$

and we can choose  $h > 0$  such that

$$\frac{h}{p}|(k+1)q(x) - q'(x)| \leq w(x) \text{ and } \frac{h}{p}|q(x)| \leq w(x) \quad \text{on } [0, 1]. \quad (4.3)$$

Since  $f_0 = \infty$ , there exists  $\delta > 0$  such that

$$|y(x)|^p < hF(y(x)) \quad \text{for } |y(x)| < \delta. \quad (4.4)$$

In this case there is a subinterval  $[0, c)$ , where  $c \in (0, 1]$ , such that  $|y(x)| < \delta$  on  $[0, c)$ . From (4.2) and (4.3), for  $x \in [0, c)$  we have

$$E[y]'(x) \leq (k+1)E[y](x).$$

i.e.,

$$E[y](x) \leq E[y](0)e^{(k+1)x} \quad \text{on } [0, c).$$

But  $E[y](0) = 0$ , and so  $E[y](x) \leq 0$  for  $x \in [0, c)$ . In fact, if there exists  $x_1 \in (0, c)$  such that  $y(x_1) \neq 0$ , then by (4.3) and (4.4) we have  $E[y](x_1) > 0$ . This leads a contradiction.

(b) Let  $f_0 < \infty$ . For the above  $\delta > 0$ , there exists  $d_1 > 0$  such that  $|f(y)| \leq d_1|y(x)|^{p-1}$  for  $|y(x)| \leq \delta$ . Note that  $|y(x)| < \delta$  on  $[0, c)$ . Applying the similar arguments in the proof of Proposition 2.1, by (2.7)-(2.8), we have

$$|y(x)|^p + |z(x)|^{p^*} \leq d(c) \int_0^x (|y(t)|^p + |z(t)|^{p^*}) dt,$$

where  $d(c)$  is some positive constant. By Gronwall inequality, it completes this case.

(iii) Let  $\eta_1 = 0$  and  $\eta_2 \neq 0$ . Without loss of generality, say  $\eta_2 = 1$ . Then

$$\frac{1}{2}|x - 0| < |y(x)| < 2|x - 0| \quad \text{near } 0. \quad (4.5)$$

Now we assume that  $y_1(x)$  and  $y_2(x)$  are two local solution of the IVP with the same initial condition. Then, by (2.2) we have

$$\begin{aligned} y_1'(x)^{(p-1)} - y_2'(x)^{(p-1)} &= (p-1) \int_0^x q(t)[y_1(t)^{(p-1)} - y_2(t)^{(p-1)}] dt \\ &+ (p-1) \int_0^x w(t)[f(y_2(t)) - f(y_1(t))] dt. \end{aligned} \quad (4.6)$$

Apply an application of the mean value theorem: for  $a_1$  and  $a_2$  of the same sign,

$$a_1^{(p-1)} - a_2^{(p-1)} = (p-1)(a_1 - a_2)|\bar{a}|^{p-2}, \quad (4.7)$$

where  $\bar{a}$  lies between  $a_1, a_2$ . Let  $W(x) = y_1(x) - y_2(x)$ .

By (4.5) and (4.7), we obtain that

$$\begin{aligned} \int_0^x |q(t)[y_1(t)^{(p-1)} - y_2(t)^{(p-1)}]| dt &\leq (p-1)\|q\|_x \int_0^x |y_1(t) - y_2(t)| |\bar{b}_t|^{p-2} dt \\ &\leq 2(p-1)\|q\|_x \int_0^x |W(t)| |t|^{p-2} dt \\ &\leq 2(p-1)\|q\|_x \|W\|_x \int_0^x |t|^{p-2} dt, \end{aligned}$$

where  $\bar{b}_t$  lies between  $y_1(t)$  and  $y_2(t)$  and recall that  $\|\cdot\|_x$  means the maximum of the given function in the interval from 0 to  $x$  and  $p-2 > -1$ . It follows the locally Lipschitz continuity of  $f$ , and then

$$\int_0^x |w(t)[f(y_2(t)) - f(y_1(t))]| dt \leq A\|W\|_x$$

for some positive constant  $A$ . So from (4.6), we have that

$$(p-1)|\bar{a}_x|^{p-2}|W'(x)| \leq B\|W\|_x,$$

where  $\bar{a}_x$  is close to  $\eta_2$  and  $B$  is some positive constant. That is,  $|W'(x)| \leq C\|W\|_x$  near 0, for some positive  $C$ . By Lemma A, we have  $W = 0$  in a neighborhood of zero.

- (iv) Let  $\eta_1 \neq 0$  and  $\eta_2 = 0$ . If  $1 < p \leq 2$ , the right hand side of (2.2) is locally Lipschitz continuous. So the local uniqueness is obvious. Let  $p > 2$ . Recall the phase and radius function of the Prüfer substitution satisfying (3.2) and (3.3). Note that the radius function  $r(x) > 0$  is uniquely defined, while the phase function  $\theta(x)$  is unique modulo  $2\pi_p$ . Since the right hand side of (3.2) is Lipschitz in  $\theta$  for  $(x, \theta) \in [0, c] \times [\pi_p/2, \phi]$  for some angle  $\phi$ ,  $\theta(x)$  is unique near  $\pi_p/2$ . Hence, the radius function  $r(x)$  is also unique. This implies that  $y$  is unique locally.

By Proposition 2.1, the solution  $y(x; \eta_1, \eta_2)$  of the IVP (1.1) and (2.1) is unique on  $[0, 1]$ . Finally, a general theory of the continuous dependence of the solutions on initial conditions [17, Chap. V, Theorem 2.1] implies that  $y(x; \eta_1, \eta_2)$  and  $y'(x; \eta_1, \eta_2)$  are continuous in  $(x; \eta_1, \eta_2) \in [0, 1] \times \mathbb{R}^2$  (see also [13]).

## Acknowledgments

The first author is supported in part by National Science Council, Taiwan under contract number NSC 99-2115-M-022-001. The author Y.H. Cheng is supported by National Science Council, Taiwan under contract numbers NSC 98-2115-M-007-008-MY3.

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